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# The Phenomenotechnical, Math education and the Experience of "Lived Number"

**Norm Friesen & Krista Francis-Poscente**

*Abstract*

Over the last half century, in considering possible directions for a future study of media, a sociology of science, or for phenomenology itself, theorists have invoked Bachelard's notion of "phenomenotechnics" (Latour & Woolgar 1980; Waldenfels 2004; Castelao-Lawless 1995). Phenomenotechnics refers to the study of "a reality [which] takes on the appearance of a phenomenon by virtue of its construction through material techniques" (Latour & Wolgar 1980). Whether these material techniques are laboratory instruments and practices, cinematic technologies, or online simulations, the effects of such "technologies of attention" as Waldenfels calls them, can be seen as "belong[ing] to the field of the phenomenotechnical, which is always already a part of the organization of experience" (2004; p. 120). In defining and undertaking a phenomenotechnical investigation, this study focuses on the use and history of one particular "technology of attention" (Waldenfels 2004), the famous "Tower of Hanoi" puzzle. Considered a classical problem in both programming and artificial intelligence, this simple "game" is widely available on the Web as a simulation, and is occasionally used in this and other forms for mathematics education. Through a careful examination of the experiential dimension of this particular "attentional technology," this paper develops an understanding of its educational value specifically in terms of its phenomenotechnical and experiential characteristics, pointing towards the importance of affective and embodied experience of numerical magnitude and relationship – rather than more formalized knowledge – in mathematical understanding.

## Introduction: Technologies of Attention

During 1880s and 1890s, there is said to have occurred in Europe "a generalized crisis in perception" and attention; "a sweeping reorganization of visual/auditory culture." In advancing this thesis, art historian Jonathan Crary, like Walter Benjamin before him, emphasizes the contribution of "technological forms of spectacle, display, projection, attraction, and recording" to this crisis. However, unlike Benjamin, Crary also underscores contemporaneous achievements in medicine and "psychophysics:"

> the rapid accumulation of knowledge about the workings of a fully embodied observer disclosed possible ways that vision was open to procedures of normalization, of quantifycation, of discipline. Once the empirical truth of

> vision was determined to lie in the body, vision (and similarly the other senses) could be annexed and controlled by external techniques of manipulation and stimulation. (1999; p. 12)

There are a range of technologies and corresponding physiological phenomena that illustrate this re-location of the “empirical truth” of perception to the body. Flickering cinematic images and persistence of vision are one; stereoscopic illusions and visual parallax, another; and the telephone, phonograph and radio, and the perception (or imperception) of frequencies of light and sound exploited by them, a third. These technologies and the corresponding perceptual “anomalies” expose a new picture of the observer. This is one whose perceptual and attentional powers – whose access to phenomenal and empirical evidence – are subject to exploitation and control. Thinking of the technological artefacts mentioned above (cinema, stereoscope, phonograph, etc.), these processes of control and discipline can be described as both powerful and paradoxical: subjects are almost inevitably individuated and immobilized in front of technological artifacts in order to participate in a significant if temporary attentional and perceptual “reorganization.” But at the same time as they are “locked into” the primitive “interfaces” of the telephone, stereoscope, etc., the very identity, stability and “self-presence” *as subjects* is subverted or thrown into question. Think, for example, of the “location” and “identity” implied by the reactions of observers who ducked in the onrush of early cinematic images of trains, or who (still today) mourn the fate of a beloved heroine in a melodrama. Think also of questions posed by more recent technologies and their “attentional manipulation:” Where are we, for example, when we're on a mobile phone? Or who are we when we're online playing Second Life? We are perforce on the phone or a user in front of the computer, but this is obviously not the entire answer.

As these contemporary examples indicate, this “crisis of attention” – and the subjectivity and technologies bound up with it – has been heightened rather than diminished in the decades since the late 19$^{th}$ century. As Crary points out, this is also expressed in recent studies of “economics of attention”, attempts to protect “attention data” and concern over proliferating diagnoses of “attention deficit disorder” (Crary 1999, pp. 35–38; attentiontrust.org 2006; Lanham 2006)

It is contexts of perceptual manipulation and attentional exploitation – first in the laboratory and now in the living room and elsewhere – in which the phenomenologies of Bachelard, Waldenfels, Latour and Wolgar and others propose a program of “phenomenotechnics.” This term receives its first mention in Bachelard's philosophy of science, in his emphasis on the ultimately “epistemic function” of instrumentation in scientific investigation. Writing specifically of mathematical physics, Bachelard explains how the “noumenology” (object of study) in this field implies “a phenomenotechnique in which new phenomena are not simply found, but invented, that is, thoroughly constructed” (Bachelard, as cited in Castelao-Lawless 1995, p. 315). Although it is not significantly altered or developed in Bachelards later work, this initial definition is echoed in Latour and Woogar's study of *Laboratory Life*, in which they emphasize how certain phenomena do not simply “*depend on* certain material instrumentation; rather [they] *are thoroughly constituted* by the material setting…” (p. 64; 1979; emphasis in the original).

Of course, such definitions could just as well apply to the use of “epistemic” technologies in science as to technological devices, manipulation and sensory illusion generally. It is in this later, broader sense that phenomenotechnique has been taken up most recently in the work of Bernhard Waldenfels. In addition to providing a number of extended discussions of this particular subject (e.g. 1998, 214ff; 2002, 374ff), Waldenfels also outlines a related methodology of the 'reality-experiment' which he says is to be undertaken through the use of “new media and technologies” (1998; p. 214). But it is his most recent articulation of phenomenotechnics in *Phänomenologie der Aufmerksamkeit* (2002; *Phenomenology of Attention*) that is most important here – specifically, Waldenfels' discussion of this subject in terms of places, moments and other “instances of confluence and “between-ness” (or *Zwischeninstanzen*; translated here as “interstitial entities”). These are instances or places “betwixt and between” – at the point(s) of contact of – “subject and object” (p. 116) “internal attitude” and “external effect” (p. 18), or “formal approach and factual substance” (2004, p. 18; 1998, p. 215). “Interstitial entities,” as Waldenfels explains, “have their sufficient grounding neither in an order of things nor in the realm of the mind;” (p. 25) they instead become manifest in a “region of the between that can neither be reduced to subjective attention nor to objective mechanism” (pp. 25, 121). Waldenfels' descriptions indicate that interstitial entities are identified by terms like “atmosphere,” “ambience,” “engagement,” “interface” “involvement” and “interaction” They are also the locus of “attentional technologies” and “media of attention” and indeed, of attention itself. Finally, they are also the noumena, the field of study proper to, the phenomenotechnical.

> Technologies of attention belong in the field of the phenomenotechnological, in which the organization of attention is always already involved. The *logos* of the phenomenon is never free from a simultaneous *techne* of phenomena, and in both cases, it is worth asking about the relationship to *pathos*, which specially proclaims itself in that which attracts our attention. (2004; p. 120)

This rich locus is the subject of study here. This study involves object, subject, passion and also history. It is rich with incident. It further involves a particular technology of attention that has its modern origin in the 19th century, precisely during the “crisis of attention” identified by Crary, Benjamin and others.

## Tower of Hanoi

Our story begins, however, not in the 19th century, but in 1550, when the Italian mathematician Girolamo Cardano is said to have written the following about the then mysterious lands of the Far East:

> A monastery in Hanoi has a golden board with three wooden pegs on it. The first of the pegs holds sixty-four gold discs in descending order of size – the largest at the bottom, the smallest at the top. The monks have orders from God to move all the discs to the third peg while keeping them in descending

> order, one at a time. A larger disc must never sit on a smaller one. All three pegs can be used. When the monks move the last disc, the world would end. (Danesi 2004, p. 109f)

Other versions of this story speak of three holy places where the discs can be kept; still others talk of diamond-tipped poles. But in each case, this sacred duty involves the measurement of an apparently unimaginable, immeasurable period of time. It is, in part, about the process of breaking into discrete tasks and moments an interval of time that – because of its enormous size – is difficult, perhaps impossible to comprehend.

The Tower enters modern history in 1883, with mathematician François Anatole Lucas. This was the year that Lucas introduced to the market a simplified, abstracted version of the "towers" in the form of a "casse-tête" (Lucas 2007), a puzzle, brain teaser or literally, "head breaker." Similar to but simpler than the situation described by Cardano, Lucas' version has three pegs and only eight discs. But the same rules and restrictions apply. Using a vaguely Asiatic, anagrammic pseudonym, Dr. "Lucas D'Amiens" promoted the puzzle as follows: "Amusing and instructive, easy to learn and to play in town, in the country, or on a voyage, it has for its aim the popularization of science, like all the other curious and novel games of professor N. CLAUS (OF SIAM)" (Lucas, as cited in Stockmeyer 1998). Referencing the original legend with 64 discs, and clearly aware of a precise exponential relationship between the numbers of discs and of the moves required, Lucas offered his customers this challenge:

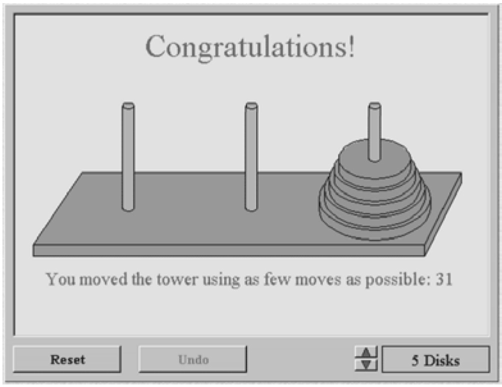
> We can offer a prize of ten thousand francs, of a hundred thousand francs, of a million francs, and more, to anyone who accomplishes, by hand, the moving of the Tower of Hanoi with sixty-four levels, following the rules of the game. (Lucas, as cited in Stockmeyer 1998)

Underneath, he knowingly added: "We will say immediately that it would be necessary to perform successively a number of moves equal to 18 446 744 073 709 551 615 which would require more than five billion centuries!" (Lucas, as cited in Stockmeyer 1998)

## "Virtual" Towers

Now, in the early $21^{st}$ century, Lucas' (re)invention, as will be discussed below, stands as a kind of 'textbook' problem in computing science and artificial intelligence. Additionally, in psychology, psychiatry and neuropsychology, the Tower of Hanoi puzzle has served as "a well-established test of executive [mental] functions," used as a kind of paradigmatic "task" in measures of attentional and problem-solving efficacy (Kopecky 2005). It is used as a control to test the effects of variables such as age, "divided attention", mental disorders and other conditions. Also, computerized versions of this puzzle, like the one shown below, abound on the World Wide Web. This is where I first discovered the Tower of Hanoi.[1]

[1] The description that follows (and the other descriptive writing presented subsequently in this paper) have been developed using the hermeneutic-phenomenological methods described by van Manen in *Researching lived experience*. London, ON: Althouse Press (1990). See also: www.phenomenologyonline.com.




*"Towers of Hanoi"*
*From the National Library of Virtual Manipulatives*
*http://nlvm.usu.edu/en/nav/frames_asid_118_g_3_t_2.html*

The game, like the one shown above, initially presented seven purple discs stacked on left-most peg. At first glance, the puzzle looked quite innocuous. Within moments I was absorbed in the task of trying to move the discs to another peg. I had to stop and think: "Okay. I want to move the next big disc over. That means I need to first stack the others in the middle." But moving the discs to accomplish this interim goal was difficult enough, and I was soon absorbed in this task alone. I felt like a dog chasing its tail as I tried to get the discs back in order. Still, the largest discs remained trapped in their original resting places, immobile. Unable to bear the aggravation, I reset the game to 3 discs.

Sophia my youngest daughter looked at the puzzle briefly as she came by, showing little interest. She then started playing the piano right beside me. She played 'A Little Song' beautifully. I usually relish in her playing, but I was feeling slightly annoyed. I wanted to concentrate and the music was distracting.

"Sophia, do you have to play that piano now?" She stopped and disappeared from the room.

I was quickly absorbed in the Tower again. I accidentally moved the wrong disc and the machine counted my corrected move. That was unfair! I reset the puzzle and the pile of discs instantly appeared on the left peg. After a couple of attempts, I was able to move the 3 disc Tower in 15 moves, the fewest possible moves.

> With one quick click of the mouse, I reset the game to four discs in order to challenge my new-found skills. I was able to move the Tower, but not efficiently. The game flashed, “Congratulations! You moved the Tower in 39 moves. The Tower can be moved in fewer moves.” The twinge behind my forehead was now a full blown ache. I was ticked. “Darn it! Where did I go wrong?”

I work with these kinds of puzzles everyday. I dismiss puzzles that are solved easily without much effort. The puzzles that capture my thoughts and drive me crazy are always my favourites. I become very emotionally attached to these puzzles; and solving one for the first time makes me feel ecstatic.

The puzzles that torture me the most are the ones I promote with teachers and students. I visit school classrooms and use puzzles to engage both students and teachers alike. In these contexts, we use “manipulables” or “manipulatives.” By this, we mean objects that can be flipped, slid and turned about, and are designed in order to help students understand mathematical abstractions. These “manipulatives” can range from popsicle sticks and building blocks to an abacus or a “physical” version of puzzles like the Tower of Hanoi. An online puzzle like the one I tried above is generally called a “computer” or “virtual” manipulative, which have been defined as

> computer programs that allow the user to manipulate representations of concrete objects, such as base-ten blocks or geoboards, on a computer screen. In addition to computerized versions of concrete manipulatives, their examples of these “computer manipulatives” also include spreadsheets, databases, and Logo-items which have never functioned either traditionally or physically as “manipulatives.” (Moyer, Bolyard, & Spikell 2002, p. 372)

When defined very broadly, “educational” manipulatives appear to have been in use for quite some time. Gerbert d’Aurillac (950–1003), a noted scholar and teacher, relied heavily on the use of such “teaching aids” which he designed and constructed to meet his students' learning needs. It is said that he “broke with all tradition by devising charts, models and instruments for demonstration to his students and for handling by them…” (Lattin 1961). For example, Gerbert is well known for his revival of the abacus for increasing the speed and accuracy of arithmetical problem solving.

But what do arrangements of toothpicks, blocks or discs and pegs have to do with mathematics and equations? Is there math in this simple but enigmatic puzzle like the Tower of Hanoi? If so, precisely *where* are the mathematics in this game?

## The Tower and Mathematics

In a text that is today considered foundational for a new and natural-scientific understanding learning, Bransford, Brown and Cocking (1999) describe the connection between the manipulable and concrete, and the abstract and mathematical in terms of what they call “progressive formalization”

> There are interesting new approaches to the development of curricula that support learning with understanding and encourage sense making. One is "progressive formalization," which begins with the informal ideas that students bring to school and gradually helps them see how these ideas can be transformed and formalized. Instructional units encourage students to build on their informal ideas in a gradual but structured manner so that they acquire the concepts and procedures of a discipline.(p. 137)

In the case of the Tower of Hanoi, the informal ideas that students bring to the puzzle would be those about rules, order, and manipulation, and the way that objects behave and games are played. The "Tower" activity should accordingly allow students to progressively build on such "informal ideas in a gradual but structured manner" so that they can be formalized – presumably and ultimately in the form of mathematical notations and expressions. (Others have used complimentary terms such as "progressive idealization" and "concreteness fading" in speaking of similar learning processes [e.g., Goldstone & Son 2005]).

Another way of understanding the mathematical relevance of the puzzle is provided by Kirkpatrick, Swafford & Findell in their 2001 report called "Adding It Up: Helping Children Learn Mathematics." In this document, they identify a number of "competency strands" that together are said to constitute "mathematical proficiency:"

- *Conceptual understanding* – comprehension of mathematical concepts, operations, and relations
- *Procedural fluency* – skill in carrying out procedures, flexibility, accurately, efficiently, and appropriately
- *Strategic competence* – ability to formulate, represent, and solve mathematical problems
- *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification
- *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's efficacy (p. 116)

The Tower of Hanoi puzzle might be seen to align with at least a few of these competency strands. It asks that students comprehend concepts, operations and relations (through not strictly mathematical ones), for example. It also asks that its users be "fluent" with a particular procedure, obeying certain rules and achieving a particular outcome, and it is certainly an exercise in "adaptive reasoning" or logical thinking (although it does not explicitly require explanation or justification from those who engage with it). Finally, completing the Tower of Hanoi successfully might indeed cultivate a belief or disposition in one's own efficacy – at least as far as solving puzzles goes.

But how is this related to the experience of engaging with the Tower of Hanoi? What is it about this puzzle that is so absorbing? Why does this it remain in existence? Why is it continually reinvented?

## It made my mind angry

> Dana started to play with the Tower puzzle on the computer. She sat with slouched shoulders, chin jutting out, staring intently at the screen. The disc wavered as she moved the mouse tentatively. Once in awhile, she had difficulty getting the disc to hold on the intended peg, but she soon got the hang of it. After several minutes she muttered “dang, I am going in circles.” Then a few moments later she asks, “I’m not doing great, what is the high score?
>
> “The point of the game is not only to move all the discs to another peg, but to do it in as few as moves as possible,” I explained. She finished in 109 moves.
>
> Unhesitating, she clicked on the blue arrows and changed the game to just 2 discs. She moved both discs in 3 moves. When the game congratulated her for moving the Tower in the minimum number of moves, she clapped her hands and exclaimed “Yeah! Yeah! Sweet!” She quickly worked her way up to 4 discs in the remaining time I was with her. She celebrated each achievement with as much enthusiasm.
>
> Whenever the game told her she could have done it less moves, she exclaimed “Dang!” or “Oh Crap!”. Each time she determinedly started again. As I observed her having difficulty, I found myself wanting to offer unsolicited advice; when she succeeded, I found myself muttering sounds of agreement.
>
> “Please stop saying that,” she asked me loudly, and then she turned back to the computer.
>
> I left her alone for awhile. She had successfully worked through moving 5 discs. However, she made a mistake while trying to pile them back on the 6$^{th}$ disc. Then a look of determination crossed her face and she restacked the puzzle. She repeated this process for 45 minutes.
>
> When I heard her cry “Sweet!”, I returned and I asked her how many tries that took. “Billions of tries” she responded. I asked what she thought of the game. “Fun” was all she said. I asked if time passed quickly. She said “no, it stood still. It felt like I had been playing for hours when I know it wasn’t that long.”
>
> I asked what she thought of playing a game that was at least 100 years old, maybe even centuries old. She said that people were nuts back then too and liked to torture themselves. “The Tower of Hanoi made her mind angry,” she added.

What was it about the Tower that attracted us both? At first, it all appears quite innocuous and simple. The rules are very basic: there are only three pegs, about the a few discs, and a tally of one's moves. But gradually, Tower of Hanoi draws those who engage with it into a strange world: It is one where, as Dana says, time stands still. It feels like hours, when you know it isn't that long. At the same time, the task being undertaken is very repetitive. Even when the puzzle is simplified to far fewer than 64 discs, it seems to take so many moves that they are impossible to enumerate: Dana makes "billions of tries," and earlier, I myself was unable to see how I could arrive at a solution in fewer moves. The experiential world of this puzzle is also one that is sharply delineated from what is happening outside, whether it be music or even supportive feedback. In both cases, those nearby are harshly silenced.

Finally, the world opened up by the Tower of Hanoi also presents a strange mixture of intellect and emotion. As it is understood in mathematical, psychological and popular terms, the Tower of Hanoi is about the head, the brain, or the intellect – not the heart, the body or the emotions. Lucas called the puzzle a "*head* breaker," a "*brain* teaser" something which "has for its aim the popularization of science." In this sense, it is presented as a formalized, thoroughly cerebral undertaking. As a "manipulative," the Tower of Hanoi can *indeed* be seen as enabling a progressive, gradually increasing formalization in the learner's understanding. The number of one's moves are counted, and sometimes with some adult encouragement (and certainly through the feedback offered by the virtual manipulative), users are led to strategize about to complete the puzzle in the smallest number of moves.

But of course, this is not the whole story. Dana's description of puzzle as making her "mind angry" both confirms the cerebral nature of the puzzle, but simultaneously points beyond it – to the somatic, the realm of the body, the heart, and the emotions. Elation and frustration are perhaps the most common feelings associated with the puzzle: Both Dana and I felt and expressed elation after our initial success. I've seen children's eyes well up with tears as they repeatedly run into difficulty, and then later, suddenly light up when they finally find a solution. Although a few people – especially adults – initially refuse to try the puzzle, once someone is engaged with it, their frustration generally does not seem to stop them from continuing and from trying again (and again). However rather than be satisfied with our success, we seem to seek more frustration, making the puzzle harder by adding more discs. Any sense of elation passes quickly as the Tower lures us to continue.

As indicated above, the role of the puzzle in scholarship is rich and complex. One could say that the Tower of Hanoi is a puzzle that has launched a thousand research projects. Given the popularity and longevity of the puzzle, it is not surprising that it has effectively produced its own tiny but thriving "mathematics" and carved out its own corners incomputational science and artificial intelligence, Strategems, algorithms, computer programs, and discussions of the complexities of "artificially intelligent" planning and problem solving – all addressing the Tower of Hanoi in one form or another – are all easy to find. Speaking specifically of computer programming, Stockmeyer writes back in 1984,

> The Tower of Hanoi puzzle … has been undergoing a dramatic revival in popularity during the past years largely due to its use as a programming exercise in elementary computer courses. Many variations on the original puzzle also have been proposed and solved.

Here's one example of a programming exercise – an “algorithm” expressed in a human rather than computer language to solve the puzzle:

1. Move the smallest disc to the peg it has not recently come from
2. Move another disc legally (there will only be one possibility)
3. Repeat.

Here's another example of an algorithm written in a programming language known as “Haskell” that calculates a list of all the moves required for the Tower of Hanoi:

```
hanoi n = hanoi' n 1 2 3
hanoi' 0 _ _ _ = []
hanoi' n f i t = (hanoi' (n-1) f t i) ++ (f, t) : (hanoi' (n-1) i f t)
```

These algorithms and representations can, in some senses, be understood as the “formalizations” that students might be progressing towards in the process of progressive formalization as described by Bransford, Brown and Cocking. The space between these formalizations and unformalized concreteness Tower puzzle itself might well be where the mathematical competencies might lie.

But this is obviously not all there is to the puzzle nor to the question of mathematical competency. Mathematical and computational abstractions or the Towers' use as an exercise for progressive formalization – or its solution in the form of mathematical and computational abstraction – do not begin to exhaust its significance.

## Exponential Experience

After Dana had mastered 6 discs she wanted to try seven discs, I prompted her to predict how many moves that might take her. I wrote a list of the number of discs and corresponding numbers of moves.

| | | |
|---|---|---|
| 2 discs | → | 3 moves |
| 3 discs | → | 7 moves |
| 4 discs | → | 15 moves |
| 5 discs | → | 31 moves |

Then I asked her how many moves would it take for six discs. She looked at the chart for just a minute or two and said, “63 moves”. Astonished, I asked “how did you figure that out?” As she pointed to the number of moves she said “the rule is itself plus itself plus 1.” It increases consistently, in other words, by a factor of 2. Dana, in other words, had stated in simplified form an iterative algorithm similar to the ‘Haskell’ algorithm above.

I had been struggling myself to come to my own understanding of the puzzle. As a veteran of hundreds of puzzles, and a researcher in mathematics education, I knew there was almost certainly a beguilingly concise and simple formula that would explain my observations and predict the number of moves for *any* number of discs. In seeking some kind of a pattern that would give me a clue, I thought long and hard about the numbers of discs and the corresponding tallies of moves.

With each disc added, the number of moves increases from 1 to 3 to 7, 15, and 31... When I added 1 to each of these numbers, I saw what I needed to see: 2, 4, 8, 16, 32. The new sequence gave me shivers up my spine. 2 times 2 equals four, 2 times 2 times 2 equals 8, 2 multiplied by itself four times ($2^4$) equals 16. The number of discs is how many times you multiply 2 by itself (with just 1 subtracted at the end). How simple, how elegant! Or to use Dana's term how "sweet!"

In mathematical terms, I had just uncovered a geometric progression: a numerical sequence in which each term is multiplied by a constant in order to obtain the next term. Mathematically, the Tower is a model geometric progression that increases by multiples of 2. As indicated in the illustration below, to get the number of moves required, each disc on the Tower means 'multiply by 2' (and then subtract one from the result).

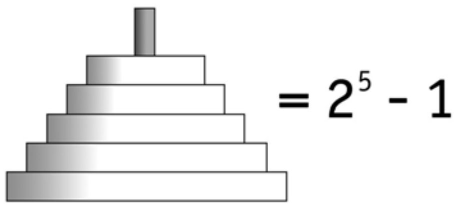



*Diagram courtesy of Krista Francis-Poscente*

But of course, this does not simply explain the puzzle away. The experience of the Tower of Hanoi is not just a matter of quantities and numerical magnitude. Each attempt to solve the puzzle is itself an experiential event. Sometimes it is an ordeal, a cause of vexation, and less often, it is an occasion for relief or even elation and celebration. This experiential event can be one of tears of agony or cries of joy – or sometimes, an alternation between the two. In this context, the difference in magnitude, say, between 22 and 26 is not just an indifferent, abstract quantity. Instead, it is experienced as "lived time" in a seemingly endless sequence of moves that two discs would require as compared to six. This experiential temporality that, as Dana observed, is one in which time "stands still", in which minutes or seconds seem like hours. This is further reinforced in descriptions of the puzzle's monastic origins as a way of measuring an apparently unimaginable, immeasurable period of time – of breaking into discrete tasks and moments an immense interval of time that is difficult, perhaps impossible, to comprehend. It is also experienced as "lived space" in terms of the highly simplified "world" created by the pegs and discs of the puzzle. This is a space, as observed above, where a set of simple rules can render some parts of the puzzle apparently immobile, impossible to release from one peg, and to get over to the other. This movement from left to right often proves much more difficult than the actual, trivial physical distance separating one pole from another would suggest.

What may make this experiential world all the more significant pedagogically is the *kind* of mathematical relationship to which it gives such vivid experiential illustration. The explicit rules governing the Tower of Hanoi are on their own linear and additive (three pegs, six or fewer discs ordered by size), but the mathematics relevant to its experience are emphatically *not* linear and additive. This mathematics is instead a geometric or exponential mathematics that is given illustration, for example, in graphs showing the familiar curve of a parabola. It is also referenced in measures of economic “growth rates” or of totals produced through compounded interest. But as we know from these and other contexts, this geometric relationship involves surprises or a kind of “wildness” that is not present in linear or even in variable relationships or patterns. Think of bacteria reproducing in the environment of a Petri dish, doubling every few hours, eventually exhausting the limited resources of this environment, and brining to an end this geometric progression. It is this relationship that is “lived” in a very different way in the puzzle, in terms of the number of discs and the “billions” of (more) moves required each time a disc is added. And it is the lived experience of this relation that makes an indispensable contribution to the puzzle's value as a mathematics “manipulative.”

Given the way in which this lived experience arises – and the degree to which it can be charged with emotion – it might be worth also calling this puzzle an emotional test, a “experiencable” rather than just a brain teaser or a manipulative. It is through this emotionally-charged, lived space and time that one could say the puzzle almost produces a sense of “lived number.” Thinking through the significant consequences of these terms and suggestions is a task that may well have implications for curricula and even theories of learning, changing an exclusive emphasis on the brain and the cognitive to an emphasis that also recognizes the somatic and the emotional. But that is the subject of another paper.

## Conclusion

At the outset of this paper, issues of attention, attentional technologies, and the question of phenomenotechnique generally were introduced in a particular historical context. The intervening phenomenological study of the Tower of Hanoi as a mathematics manipulative, as a brief but intense lived experience of geometric relation, has hopefully provided some experiential terms in which these issues and this context can be understood. In this sense, the Tower of Hanoi represents a relatively “primitive” kind of attentional mechanism, appearing contemporaneously with the general “crisis” or “reorganization of experience” occurring in the 1880's and 1890's. As a kind of rudimentary phenomenotechnique, as a simple but powerful “interstitial entity,” the Tower of Hanoi is a technology of attention that, in its own way, seems no less controlling and powerful than that of the cinema or the stereoscope. But unlike these technologies, it does not gain its power by explicitly “annexing,” “exploiting,” or “controlling” any single identifiable perceptual anomaly or limitation on the part of its human users. As is captured in the term “manipulative,” the Tower of Hanoi is not a material technique that reproduces visual or aural experience (unlike the stereogram or phonograph). Instead, it is one that entails kinaesthetic involvement in a realm that is relatively abstract in its signification. When it is combined with Web and computer technology, the Tower of Hanoi can be said to be made it into a kind of combined phenomenotechnique: the “user illusion" of the computer's visual-kinaesthetic interface (Kay 1990) is combined with the organization of experience provided by the puzzle, resulting in a compounded effect.

In the present historical situation, the Tower of Hanoi has acquired a kind of exemplary status as an attentional heuristic, a kind of simulator or stimulator of cerebration – in psychology, psychiatry, neuropsychology, programming, artificial intelligence, and in a slightly different way, in education and mathematics. In an era where attention is substantivized as a commodity, as subject to economic laws of value and scarcity, the Tower of Hanoi has come to represent a kind of generic or "content free" manner of controlling, measuring and consuming this valuable mental and psychological resource. But when understood in this way, this phenomenotechnique is in a sense "black boxed," effectively rendering the experience and substance implied by it unproblematic or invisible. By implication, questions concerning attention *itself* are also foreclosed. Attention is reduced to the objective form of the puzzle – or of any other material means of constructing, directing and controlling attention – with subjective elements being reduced to the form of uncontrolled variables. In emphasizing the inseparable interconnection of subject and object in the form of *Zwischeninstanzen* or interstitial entities, studying attention and attentional technologies in terms of the phenomenotechnical presents a valuable alternative perspective on this vital subject. Not only does it bring to the fore the affective and somatic aspects of phenomena that are generally understood as purely cognitive and cerebral, it also serves as a corrective to the present-day "preponderance of the object" (Adorno 1973; p. 192).

## References

Adorno, T. (1973) *Negative dialectics*. London: Routledge.

Bransford, J.D., Brown, A.L., & Cocking, R.R. (Eds.). (1999) *How people learn: Brain, mind, experience and school*. Washington, D.C.: National Academy Press.

Castelao-Lawless, T. (1995) *Phenomenotechnique in historical perspective: Its origins and implications for philosophy of science*. Philosophy of Science 62(1) pp. 44–59.

Crary, J. (1999) *Suspensions of perception: attention, spectacle, and modern culture*. Cambridge, MA, and London: MIT Press,.

Danesi, M. (2004) *The liars paradox and the Towers of Hanoi: The ten greatest puzzles of all time*. New Jersey: John Wiley.

Goldstone, R.L. & Son, J.Y. (2005) *The transfer of scientific principles using concrete and idealized simulations*. Journal of the Learning Sciences.(1)14, pp. 69–110.

Iskold, A. (2007) *The attention economy: An overview*. Available from: <http://www.readwriteweb.com/archives/attention_economy_overview.php> [Accessed August 8, 2007].

Kay, A. (1990) User interface: A personal view. In *The art of human-computer interface design*., ed. B. Laurel. Reading, MA: Addison-Wesley, pp. 191–207.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kopecky, H., Chang, H., Klorman, T., Thatcher, J.E., & Borgstedt, A.E. (2005) *Performance and private speech of children with Attention-Deficit/Hyperactivity disorder while taking the Tower of Hanoi test: effects of depth of search, diagnostic subtype, and methylphenidate*. Journal of Abnormal Child Psychology 33 (5) pp. 625–638.

Lanham, R.A. (2006) *The economics of attention: Style and substance in the age of information*. Chicago: University Of Chicago Press.

Lattin, H.P. (1961) *The letters of Gerbert: with his papal privileges of Sylvester II*. New York, NY: Columbia University Press.

Latour, B. & Woolgar, S. (1979) *Laboratory life*. Beverly Hills: Sage. Laudan, L.

Moyer, P.S., Bolyard, J.J., & Spikell, M.A. (2002) *What are virtual manipulatives?* Teaching Children Mathematics 8 (6) pp. 372–377.

From the National Library of Virtual Manipulatives. (2007) *"Towers of Hanoi"* Available from: <http://nlvm.usu.edu/en/nav/frames_asid_118_g_3_t_2.html> [Accessed August 8, 2007].

Waldenfels, B. (2004) *Phänomenologie der Aufmerksamkeit*. Frankfurt: Suhrkamp.

Waldenfels, B. (2002) *Bruchlinien der Erfahrung*. Frankfurt: Suhrkamp.